

Inequalities Among Logarithmic-Mean Measures

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Abstract

In this paper we shall consider some famous means such as arithmetic, harmonic, geometric, logarithmic means, etc. Inequalities involving logarithmic mean with differences among other means are presented.

Key words: Arithmetic mean, geometric mean, harmonic mean, logarithmic mean and inequalities.

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1 Introduction

Let us consider the following well known *mean of order t* [1]:

$$B_t(a, b) = \begin{cases} \left(\frac{a^t + b^t}{2}\right)^{1/t}, & t \neq 0 \\ \sqrt{ab}, & t = 0 \\ \max\{a, b\}, & t = \infty \\ \min\{a, b\}, & t = -\infty \end{cases} \quad (1.1)$$

for all $a, b, t \in \mathbb{R}$, $a, b > 0$.

In particular, we have

$$\begin{aligned} B_{-1}(a, b) &= H(a, b) = \frac{2ab}{a+b}, \\ B_0(a, b) &= G(a, b) = \sqrt{ab}, \\ B_{1/2}(a, b) &= N_1(a, b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2, \\ B_1(a, b) &= A(a, b) = \frac{a+b}{2}, \end{aligned}$$

and

$$B_2(a, b) = S(a, b) = \sqrt{\frac{a^2 + b^2}{2}}.$$

The means, $H(a, b)$, $G(a, b)$, $A(a, b)$ and $S(a, b)$ are known in the literature as *harmonic*, *geometric*, *arithmetic* and *root-square means* respectively. For simplicity we can call the measure, $N_1(a, b)$ as *square-root mean*. It is well know that [1] the *mean of order t* given in (1.1) is monotonically increasing in t , then we can write

$$H(a, b) \leq G(a, b) \leq N_1(a, b) \leq A(a, b) \leq S(a, b). \quad (1.2)$$

Dragomir and Pearce [2] proved the following inequality:

$$\frac{a^r + b^r}{2} \leq \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \leq \left(\frac{a+b}{2}\right)^r,$$

for all $a, b > 0$, $a \neq b$, $r \in (0, 1)$. In particular take $r = \frac{1}{2}$ in (1.1), we get

$$\frac{\sqrt{a} + \sqrt{b}}{2} \leq \frac{2(b^{3/2} - a^{3/2})}{3(b-a)} \leq \sqrt{\frac{a+b}{2}}, \quad a \neq b. \quad (1.3)$$

After necessary calculations in (1.3), we get

$$\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 \leq \frac{a + \sqrt{ab} + b}{3} \leq \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right) \left(\sqrt{\frac{a+b}{2}}\right). \quad (1.4)$$

On the other side we can easily check that

$$\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right) \left(\sqrt{\frac{a+b}{2}}\right) \leq \frac{a+b}{2}. \quad (1.5)$$

The expressions (1.3), (1.4) and (1.5) lead us to the following inequality:

$$H(a, b) \leq G(a, b) \leq N_1(a, b) \leq N_3(a, b) \leq N_2(a, b) \leq A(a, b) \leq S(a, b), \quad (1.6)$$

where

$$N_2(a, b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right) \left(\sqrt{\frac{a+b}{2}}\right), \quad (1.7)$$

and

$$N_3(a, b) = \frac{a + \sqrt{ab} + b}{3}. \quad (1.8)$$

Thus we have three new means, where $N_1(a, b)$ appears as a natural way. The $N_2(a, b)$ can be seen in Taneja [3], [4] and the mean $N_3(a, b)$ can be seen in Zhang and Wu [7].

Some more properties of the means given in (1.6) are studied by [8], [9]. A survey on these inequalities can be seen in [6].

Based on the inequalities (1.6), the author [4]-[5] proved the following results

$$\begin{aligned}
H(a, b) &\leq \frac{2A(a, b)H(a, b)}{A(a, b) + H(a, b)} \leq G(a, b) \leq \frac{2H(a, b) + S(a, b)}{3} \\
&\leq \frac{A(a, b) + H(a, b)}{2} \leq \sqrt{\frac{(A(a, b))^2 + (H(a, b))^2}{2}} \leq \frac{S(a, b) + G(a, b)}{2} \\
&\leq \frac{H(a, b) + 2S(a, b)}{3} \leq A(a, b) \leq S(a, b) + H(a, b) - G(a, b) \\
&\leq S(a, b) \leq 3[A(a, b) - G(a, b)] + H(a, b)
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
H(a, b) &\leq G(a, b) \leq \frac{G(a, b) + 2N_2(a, b)}{3} \leq N_1(a, b) \leq \frac{2A(a, b) + 7N_1(a, b)}{9} \\
&\leq N_2(a, b) \leq \frac{A(a, b) + N_1(a, b)}{2} \leq \frac{7A(a, b) + H(a, b)}{8} \leq A(a, b)
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
G(a, b) &\leq \frac{S(a, b) + 3G(a, b)}{4} \leq N_1(a, b) \leq \frac{S(a, b) + 8N_1(a, b)}{9} \\
&\leq N_3(a, b) \leq N_2(a, b) \leq \frac{A(a, b) + N_1(a, b)}{2} \\
&\leq \frac{S(a, b) + 2N_1(a, b)}{3} \leq \frac{S(a, b) + 4N_2(a, b)}{5} \leq A(a, b).
\end{aligned} \tag{1.11}$$

The above inequalities are based on the following two lemmas [3]:

Lemma 1.1. *Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ be a convex and differentiable function satisfying $f(1) = f'(1) = 0$. Consider a function*

$$\phi_f(a, b) = af\left(\frac{b}{a}\right), \quad b > a > 0, \tag{1.12}$$

then the function $\phi_f(a, b)$ is convex in \mathbb{R}_+^2 , and satisfies the following inequality:

$$0 \leq \phi_f(a, b) \leq \left(\frac{b-a}{a}\right) \phi'_f(a, b). \tag{1.13}$$

Lemma 1.2. *Let $f_1, f_2 : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ be two convex functions satisfying the assumptions:*

- (i) $f_1(1) = f'_1(1) = 0, f_2(1) = f'_2(1) = 0$;
- (ii) f_1 and f_2 are twice differentiable in \mathbb{R}_+ ;
- (iii) there exists the real constants α, β such that $0 \leq \alpha < \beta$ and

$$\alpha \leq \frac{f''_1(x)}{f''_2(x)} \leq \beta, \quad f''_2(x) > 0, \tag{1.14}$$

for all $x > 0$ then we have the inequalities:

$$\alpha \phi_{f_2}(a, b) \leq \phi_{f_1}(a, b) \leq \beta \phi_{f_2}(a, b), \quad (1.15)$$

for all $a, b \in (0, \infty)$.

The following inequality involving *log-mean* is also known in the literature:

$$H(a, b) \leq G(a, b) \leq L(a, b) \leq N_1(a, b), \quad (1.16)$$

where, $L(a, b) = \frac{b-a}{\ln b - \ln a}$, $b \neq a$ is the well-known *logarithmic mean*.

Finally, the expressions (1.6) and (1.16) lead us to the following inequality:

$$H(a, b) \leq G(a, b) \leq L(a, b) \leq N_1(a, b) \leq N_3(a, b) \leq N_2(a, b) \leq A(a, b) \leq S(a, b). \quad (1.17)$$

2 Difference of Means

Let us rewrite the inequalities (1.13) as

$$L(a, b) \leq N_1(a, b) \leq N_3(a, b) \leq N_2(a, b) \leq A(a, b) \leq S(a, b). \quad (2.1)$$

In view of (2.1), let us consider the following nonnegative differences

$$M_{SL}(a, b) = S(a, b) - L(a, b), \quad (2.2)$$

$$M_{AL}(a, b) = A(a, b) - L(a, b), \quad (2.3)$$

$$M_{N_2L}(a, b) = N_2(a, b) - L(a, b), \quad (2.4)$$

$$M_{N_3L}(a, b) = N_3(a, b) - L(a, b), \quad (2.5)$$

and

$$M_{N_1L}(a, b) = N_1(a, b) - L(a, b). \quad (2.6)$$

The inequalities (2.1) still admits more nonnegative differences, but they are already studied before.

Now, we shall prove the convexity of the above differences (2.2)-(2.6).

Theorem 2.1. *The means given by (2.2)-(2.6) are nonnegative and convex in \mathbb{R}_+^2 .*

We shall use Lemma 1.1 to prove the above theorem. We shall write each measure in the form of generating function, and then give their first and second order derivatives. According to Lemma 1.1 it is sufficient to show that the second order derivative in each case is nonnegative.

• **For** $M_{SL}(a, b)$: We can write

$$M_{SL}(a, b) = a f_{SL} \left(\frac{b}{a} \right), \quad a \neq b, \quad a, b > 0$$

where

$$f_{SL}(x) = \sqrt{\frac{x^2 + 1}{2}} - \frac{x - 1}{\ln x}, \quad x \neq 1.$$

Writing the first and second order derivatives of $f_{SL}(x)$, we have

$$f'_{SL}(x) = \frac{x}{\sqrt{2(x^2 + 1)}} - \frac{x \ln x - x + 1}{x (\ln x)^2}, \quad x \neq 1$$

and

$$\begin{aligned} f''_{SL}(x) &= \frac{(2x^2 + 2)^{3/2} [(x + 1) \ln x - 2(x - 1)] + 2x^2 (\ln x)^3}{x^2 (2x^2 + 2)^{3/2} (\ln x)^3} \\ &= \frac{2}{(2x^2 + 2)^{3/2}} + \frac{2}{x^2 (\ln x)^2} \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) \\ &= \frac{2}{(2x^2 + 2)^{3/2}} + k(x), \quad x \neq 1, \end{aligned} \tag{2.7}$$

where

$$k(x) = \frac{2}{x^2 (\ln x)^2} \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) = \frac{2}{x^2 (\ln x)^2} f_{AL}(x), \quad x \neq 1, \quad x > 0. \tag{2.8}$$

We can easily check that

$$k(1) = \lim_{x \rightarrow 1} k(x) = \frac{1}{6}. \tag{2.9}$$

Since $A(a, b) \geq L(a, b)$, this gives that $f''_{AL}(x) \geq 0$, $\forall x \in (0, \infty)$, $x \neq 1$. Moreover, $\lim_{x \rightarrow 1} f_{AL}(1) = \lim_{x \rightarrow 1} f'_{AL}(1) = 0$.

• **For** $M_{AL}(a, b)$: We can write

$$M_{AL}(a, b) = a f_{AL} \left(\frac{b}{a} \right), \quad a \neq b, \quad a, b > 0$$

where

$$f_{AL}(x) = \frac{x + 1}{2} - \frac{x - 1}{\ln x}, \quad x \neq 1$$

Writing the first and second order derivatives of $f_{AL}(x)$, we have

$$f'_{AL}(x) = \frac{x \ln x (\ln x - 2) + 2(x - 1)}{2x (\ln x)^2}, \quad x \neq 1,$$

and

$$f''_{AL}(x) = \frac{(x + 1) \ln x - 2(x - 1)}{x^2 (\ln x)^3} = k(x), \quad x \neq 1, x > 0, \quad (2.10)$$

Since $A(a, b) \geq L(a, b)$, this gives that $f''_{AL}(x) \geq 0, \forall x \in (0, \infty), x \neq 1$. Moreover, $\lim_{x \rightarrow 1} f_{AL}(1) = \lim_{x \rightarrow 1} f'_{AL}(1) = 0$.

• **For $M_{N_2L}(a, b)$:** We can write

$$M_{N_2L}(a, b) = a f_{N_2L} \left(\frac{b}{a} \right), \quad a \neq b, a, b > 0$$

where

$$f_{N_2L}(x) = \frac{x + \sqrt{x} + 1}{3} - \frac{x - 1}{\ln x}, \quad x \neq 1$$

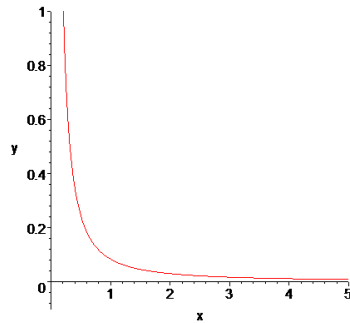
Writing the first and second order derivatives of $f_{N_2L}(x)$, we have

$$f'_{N_2L}(x) = \frac{\sqrt{x} (2\sqrt{x} + 1) (\ln x)^2 - 6 (x \ln x - (x - 1))}{6x (\ln x)^2},$$

and

$$\begin{aligned} f''_{N_2L}(x) &= \frac{12 ((x + 1) \ln x - 2(x - 1)) - \sqrt{x} (\ln x)^3}{12x^2 (\ln x)^3}. \\ &= \frac{1}{12x^2 (\ln x)^2} \left[-\sqrt{x} (\ln x)^2 + 24 \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) \right] \\ &= -\frac{1}{12x^{3/2}} + \frac{2}{x^2 (\ln x)^2} \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) \\ &= -\frac{1}{12x^{3/2}} + k(x). \end{aligned} \quad (2.11)$$

The above graph of the function $f''_{N_2L}(x)$ is given by



From the above graph we observe that the function $k_1(x)$ is nonnegative for all $x \in (0, \infty)$, and consequently, $f''_{N_2L}(x) \geq 0, \forall x \in (0, \infty), x \neq 1$.

Moreover, $\lim_{x \rightarrow 1} f_{N_2L}(1) = \lim_{x \rightarrow 1} f'_{N_2L}(1) = 0$. Also we have

$$\lim_{x \rightarrow \infty} f''_{N_2L}(x) = \lim_{x \rightarrow \infty} \left(-\frac{1}{12x^{3/2}} + k(x) \right) = \infty$$

• **For $M_{N_3L}(a, b)$:** We can write

$$M_{N_3L}(a, b) = a f_{N_3L} \left(\frac{b}{a} \right), \quad a \neq b, a, b > 0$$

where

$$f_{N_3L}(x) = \frac{(\sqrt{x} + 1) \sqrt{2(x+1)}}{4} - \frac{x-1}{\ln x}, \quad x \neq 1$$

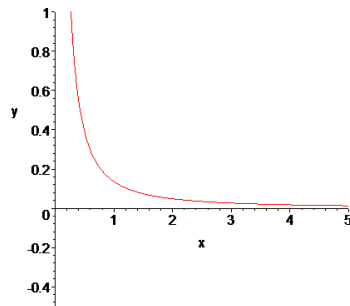
Writing the first and second order derivatives of $f_{N_3L}(x)$, we have

$$f'_{N_3L}(x) = \frac{\sqrt{x} (\ln x)^2 (2x + \sqrt{x} + 1) - 4\sqrt{2(x+1)} (x \ln x - x + 1)}{4x (\ln x)^2 \sqrt{2(x+1)}},$$

and

$$\begin{aligned} f''_{N_3L}(x) &= \frac{4(2x+2)^{3/2} ((x+1) \ln x - 2(x-1)) - \sqrt{x} (x^{3/2} + 1) (\ln x)^3}{4x^2 (\ln x)^3 (2x+2)^{3/2}} \\ &= \frac{1}{8x^2 (\ln x)^2} \left[-\sqrt{x} \left(\frac{x^{3/2} + 1}{(2x+2)^{3/2}} \right) (\ln x)^2 + 16 \left(\frac{x+1}{2} - \frac{x-1}{\ln x} \right) \right] \\ &= -\frac{(x^{3/2} + 1)}{8x^{3/2} (2x+2)^{3/2}} + k(x) \end{aligned} \quad (2.12)$$

The graph of the function $f''_{N_3L}(x)$ is given by



From the above graph we observe that the function $f''_{N_3L}(x)$ is nonnegative for all $x \in (0, \infty)$, and consequently, $f''_{N_3L}(x) \geq 0, \forall x \in (0, \infty)$.

Moreover, $\lim_{x \rightarrow 1} f_{N_3L}(1) = \lim_{x \rightarrow 1} f'_{N_3L}(1) = 0$. Also we have

$$\lim_{x \rightarrow \infty} f''_{N_3L}(x) = \lim_{x \rightarrow \infty} \left(-\frac{(x^{3/2} + 1)}{8x^{3/2}(2x + 2)^{3/2} + k(x)} \right) = \infty$$

• **For $M_{N_1L}(a, b)$:** We can write

$$M_{N_1L}(a, b) = a f_{N_1L} \left(\frac{b}{a} \right), \quad a \neq b, \quad a, b > 0,$$

where

$$f_{N_1L}(x) = \left(\frac{\sqrt{x} + 1}{2} \right)^2 - \frac{x - 1}{\ln x}, \quad x \neq 1$$

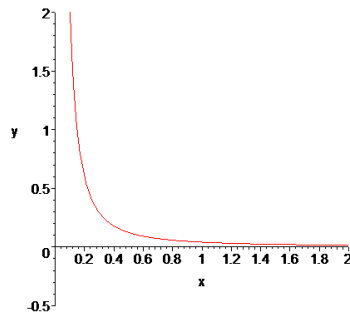
Writing the first and second order derivatives of $f_{N_1L}(x)$, we have

$$f'_{N_1L}(x) = \frac{x (\ln x)^2 (\sqrt{x} + 1) - 4\sqrt{x} (x \ln x - x + 1)}{4x^{3/2} (\ln x)^2}$$

and

$$\begin{aligned} f''_{N_1L}(x) &= \frac{8[(x + 1) \ln x - 2(x - 1)] - \sqrt{x} (\ln x)^3}{8x^2 (\ln x)^3} \\ &= \frac{1}{8x^2 (\ln x)^2} \left[-\sqrt{x} (\ln x)^2 + 16 \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) \right] \\ &= -\frac{1}{8x^{3/2}} + \frac{2}{x^2 (\ln x)^2} \left(\frac{x + 1}{2} - \frac{x - 1}{\ln x} \right) = -\frac{1}{8x^{3/2}} + k(x) \end{aligned} \quad (2.13)$$

The graph of the function $f''_{N_1L}(x)$ is given by



From the above graph we observe that the function $f''_{N_1L}(x)$ is nonnegative for all $x \in (0, \infty)$, and consequently, $f''_{N_1L}(x) \geq 0, \forall x \in (0, \infty)$.

Moreover, $\lim_{x \rightarrow 1} f_{N_1L}(1) = \lim_{x \rightarrow 1} f'_{N_1L}(1) = 0$. Also we have

$$\lim_{x \rightarrow \infty} f''_{N_1L}(x) = \lim_{x \rightarrow \infty} \left(-\frac{1}{8x^{3/2}} + k(x) \right) = \infty$$

We see that in all the cases the generating function $f_{(\cdot)}(1) = f'_{(\cdot)}(1) = 0$ and the second derivative is positive for all $x \in (0, \infty)$. This proves the *nonnegativity* and *convexity* of the means (2.2)-(2.6) in \mathbb{R}_+^2 . This completes the proof of the theorem.

3 Log-Mean Inequalities

In view of (2.1), the following inequalities are obvious

$$M_{N_1L}(a, b) \leq M_{N_3L}(a, b) \leq M_{N_2L}(a, b) \leq M_{AL}(a, b) \leq M_{SL}(a, b), \quad (3.1)$$

$$M_{N_3N_1}(a, b) \leq M_{N_2N_1}(a, b) \leq M_{AN_1}(a, b) \leq M_{SN_1}(a, b), \quad (3.2)$$

$$M_{N_2N_3}(a, b) \leq M_{AN_3}(a, b) \leq M_{SN_3}(a, b) \quad (3.3)$$

and

$$M_{AN_3}(a, b) \leq M_{SN_2}(a, b). \quad (3.4)$$

Here we shall prove an improvement over the inequalities (3.1). While, the improvement over the inequalities (3.2)-(3.4) is already given in (1.9)-(1.11). The following theorem holds:

Theorem 3.1. *The following inequalities hold:*

$$M_{SL}(a, b) \leq \frac{5}{2} M_{AL}(a, b) \leq 5 M_{N_2L}(a, b) \leq 6 M_{N_1L} \quad (3.5)$$

and

$$M_{SL}(a, b) \leq 4 M_{N_3L}(a, b) \leq 10 M_{N_1L}. \quad (3.6)$$

The proof of the above theorem is based on Lemma 1.2 and is given in parts in the following propositions.

Proposition 3.1. *We have*

$$M_{SL}(a, b) \leq \frac{5}{2} M_{AL}(a, b). \quad (3.7)$$

Proof. Let us consider

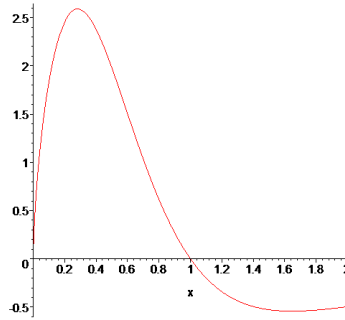
$$g_{SL-AL}(x) = \frac{f''_{SL}(x)}{f''_{AL}(x)} = \frac{2x^2(\ln x)^3 + (2 + 2x^2)^{3/2} [(x+1) \ln x - 2(x-1)]}{(2 + 2x^2)^{3/2} [(x+1) \ln x - 2(x-1)]}, \quad x \neq 1$$

for all $x \in (0, \infty)$, where $f''_{SL}(x)$ and $f''_{AL}(x)$ are as given by (2.7) and (2.10) respectively.

Calculating the first order derivative of the function $g_{SL-AL}(x)$ with respect to x , one gets

$$g'_{SH-SL}(x) = \frac{4x(\ln x)^2}{(2 + 2x^2)^{5/2} [(x+1) \ln x - 2(x-1)]^2} \times \{6(x-1)(x^2+1) - \ln x [6(x^3+1) - (x^2-x-2) \ln x] + 2x^3(\ln x)^2\}$$

The graph of the function $g'_{SH-SL}(x)$ is given by



We observe from the above graph the following:

$$g'_{SL-AL}(x) = \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (3.8)$$

Let us calculate now $g_{SL-AL}(1)$. We observe that

$$g_{SL-AL}(x)|_{x=1} = \frac{(f''_{SL}(x))'}{(f''_{AL}(x))'} \Big|_{x=1} = \frac{(f''_{SL}(x))''}{(f''_{AL}(x))''} \Big|_{x=1} = \text{indetermination.}$$

Calculating third order derivatives of numerator and denominator, we have

$$g_{SL-AL}(1) = \frac{(f''_{SL}(x))'''}{(f''_{AL}(x))'''} \Big|_{x=1} = \frac{10}{4} = \frac{5}{2}. \quad (3.9)$$

By the application of (1.14) with (3.9) we get (3.7). □

Proposition 3.2. *We have*

$$M_{AL}(a, b) \leq 2M_{N_2L}(a, b). \quad (3.10)$$

Proof. Let us consider

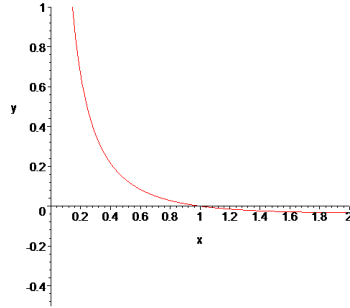
$$g_{AL-N_2L}(x) = \frac{f''_{AL}(x)}{f''_{N_2L}(x)} = \frac{12x^{3/2} [(x+1) \ln x - 2(x-1)]}{x^2(\ln x)^3 - 12x^{3/2} [(x+1) \ln x - 2(x-1)]}, \quad x \neq 1$$

for all $x \in (0, \infty)$, where $f''_{AL}(x)$ and $f''_{N_2L}(x)$ are as given by (2.10) and (2.11) respectively.

Calculating the first order derivative of the function $g_{AL-N_2L}(x)$ with respect to x , one gets

$$g'_{AL-N_2L}(x) = \frac{6x^{5/2}(\ln x)^2 \{ \ln x [(x-1) \ln x - 6(x+1)] - 12(x-1) \}}{\{x^2(\ln x)^3 - 12x^{3/2} [(x+1) \ln x - 2(x-1)]\}^2}, \quad x \neq 1$$

The graph of the function $g'_{AL-N_2L}(x)$ is given by



We observe from the above graph the following:

$$g'_{AL-N_2L}(x) = \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (3.11)$$

Let us calculate now $g_{AL-N_2L}(1)$. We observe that

$$g_{AL-N_2L}(x)|_{x=1} = \frac{(f''_{AL}(x))'}{(f''_{N_2L}(x))'} \Big|_{x=1} = \frac{(f''_{AL}(x))''}{(f''_{N_2L}(x))''} \Big|_{x=1} = \text{indetermination}$$

Calculating third order derivatives of numerator and denominator, we have

$$g_{AL-N_2L}(1) = \frac{(f''_{AL}(x))'''}{(f''_{N_2L}(x))'''} \Big|_{x=1} = \frac{-12}{-6} = 2. \quad (3.12)$$

By the application of (1.14) with (3.12) we get (3.10). \square

Proposition 3.3. *We have*

$$M_{N_2L}(a, b) \leq 2M_{N_1L}(a, b). \quad (3.13)$$

Proof. Let us consider

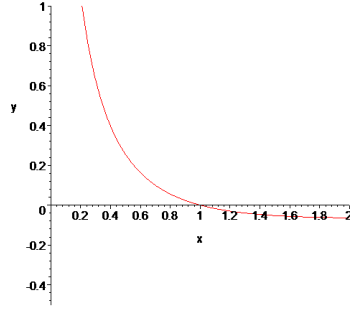
$$g_{N_2L-N_1L}(x) = \frac{f''_{N_2L}(x)}{f''_{N_1L}(x)} = \frac{2x^2(\ln x)^3 + 12x^{3/2}[2(x-1) - (x+1)\ln x]}{3x^2(\ln x)^3 + 8x^{3/2}[2(x-1) - (x+1)\ln x]}, \quad x \neq 1$$

for all $x \in (0, \infty)$, where $f''_{N_2L}(x)$ and $f''_{N_1L}(x)$ are as given by (2.11) and (2.13) respectively.

Calculating the first order derivative of the function $g_{N_2L-N_1L}(x)$ with respect to x , one gets

$$g'_{N_2L-N_1L}(x) = -\frac{4x^{5/2}(\ln x)^2 \{12(x-1) + \ln x[(x-1)\ln x - 6(x+1)]\}}{3\{x^2(\ln x)^3 + 8x^{3/2}[2(x-1) - (x+1)\ln x]\}^2}.$$

The graph of the function $g'_{N_2L-N_1L}(x)$ is given by



We observe from the above graph the following:

$$g'_{N_2L-N_1L}(x) = \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (3.14)$$

Let us calculate now $g_{N_2L-N_1L}(1)$. We observe that

$$g_{N_2L-N_1L}(x)|_{x=1} = \frac{(f''_{N_2L}(x))'|}{(f''_{N_1L}(x))'|} \Big|_{x=1} = \frac{(f''_{N_2L}(x))''}{(f''_{N_1L}(x))''} \Big|_{x=1} = \text{indetermination}$$

Calculating third order derivatives of numerator and denominator, we have

$$g_{N_2L-N_1L}(1) = \frac{(f''_{N_2L}(x))'''}{(f''_{N_1L}(x))'''} \Big|_{x=1} = \frac{-12}{-6} = 2. \quad (3.15)$$

By the application of (1.14) with (3.15) we get (3.13). \square

Proposition 3.4. *We have*

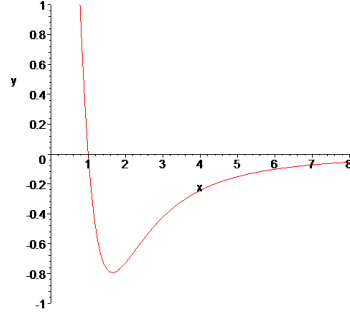
$$M_{SL}(a, b) \leq 4 M_{N_3L}(a, b). \quad (3.16)$$

Proof. Let us consider

$$\begin{aligned} g_{SL_{N_3L}}(x) &= \frac{f''_{SL}(x)}{f''_{N_3L}(x)} \\ &= \frac{-4(2x+2)^{3/2} [2x^2(\ln x)^3 + (2x^2+2)^{3/2} ((x+1)\ln x - 2(x-1))]}{(2x^2+2)^{3/2} [(x^{3/2}+1)x\ln x - 4(2x+2)^{3/2} ((x+1)\ln x - 2(x-1))]}, \quad x \neq 1. \end{aligned}$$

for all $x \in (0, \infty)$, where $f''_{SL}(x)$ and $f''_{N_3L}(x)$ are as given by (2.7) and (2.12) respectively.

Calculating the first order derivative of the function $g_{SL_{N_3L}}(x)$ with respect to x , and making necessary calculations the graph of the function $g'_{SL_{N_3L}}(x)$ is given by



We observe from the above graph the following:

$$g'_{SL_{N_3L}}(x) = \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (3.17)$$

Let us calculate now $g_{SL_{N_3L}}(1)$. We observe that

$$g_{SL_{N_3L}}(x)|_{x=1} = \frac{(f''_{SL}(x))'}{(f''_{N_3L}(x))'} \Big|_{x=1} = \frac{(f''_{SL}(x))''}{(f''_{N_3L}(x))''} \Big|_{x=1} = \text{indetermination}$$

Calculating third order derivatives of numerator and denominator, we have

$$g_{SL_{N_3L}}(x)|_{x=1} = \frac{(f''_{SL}(x))'''}{(f''_{N_3L}(x))'''} \Big|_{x=1} = \frac{-160\sqrt{2}}{-40\sqrt{2}} = 4 \quad (3.18)$$

By the application of (1.14) with (3.18) we get (3.16). \square

Proposition 3.5. *We have*

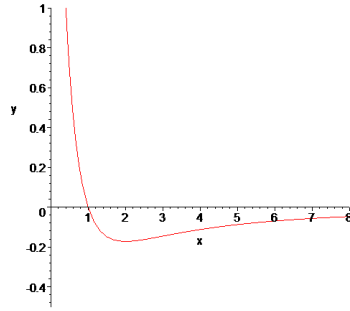
$$M_{N_3L}(a, b) \leq \frac{5}{2} M_{N_1L}(a, b). \quad (3.19)$$

Proof. Let us consider

$$\begin{aligned} g_{N_3L-N_1L}(x) &= \frac{f''_{N_3L}(x)}{f''_{N_1L}(x)} \\ &= \frac{x^2(\ln x)^3(x^{3/2} + 1) - 4x^{3/2}(2x + 2)^{3/2}((x + 1)\ln x - 2(x - 1))}{(2x + 2)^{3/2}[-x^2(\ln x)^3 + 8x^{3/2}((x + 1)\ln x - 2(x - 1))]}, \quad x \neq 1. \end{aligned}$$

for all $x \in (0, \infty)$, where $f''_{N_3L}(x)$ and $f''_{N_1L}(x)$ are as given by (2.12) and (2.13) respectively.

Calculating the first order derivative of the function $g_{N_3L-N_1L}(x)$ with respect to x and making necessary calculations the graph of the function $g'_{N_3L-N_1L}(x)$ is given by



We observe from the above graph the following:

$$g'_{N_3L-N_1L}(x) = \begin{cases} > 0, & x < 1 \\ < 0, & x > 1 \end{cases} \quad (3.20)$$

Let us calculate now $g_{N_3L-N_1L}(1)$. We observe that

$$g_{N_3L-N_1L}(x)|_{x=1} = \frac{(f''_{N_3L}(x))'}{(f''_{N_1L}(x))'} \Big|_{x=1} = \frac{(f''_{N_3L}(x))''}{(f''_{N_1L}(x))''} \Big|_{x=1} = \text{indetermination}$$

Calculating third order derivatives of numerator and denominator, we have

$$g_{N_3L-N_1L}(x)|_{x=1} = \frac{(f''_{N_3L}(x))'''}{(f''_{N_1L}(x))'''} \Big|_{x=1} = \frac{-20}{-8} = \frac{5}{2}. \quad (3.21)$$

By the application of (1.14) with (3.21) we get (3.19). □

Proof of the theorem 3.2. Propositions 3.1-3.3 together proves the inequalities (3.5) and the propositions 3.4-3.5 proves the inequalities (3.6).

4 Remarks

In this section we shall give some remarks based on the Theorem 3.1 given in section 3.

Remark 4.1. *Following the similar lines of Theorem 3.1, still we can prove that*

$$M_{SH}(a, b) \leq \frac{9}{5}M_{SL}(a, b) \quad (4.1)$$

$$M_{AG}(a, b) \leq \frac{3}{2}M_{AL}(a, b) \quad (4.2)$$

and

$$M_{SN_1}(a, b) \leq \frac{9}{10}M_{SL}(a, b), \quad (4.3)$$

where

$$M_{SH}(a, b) = S(a, b) - H(a, b)$$

$$M_{AG}(a, b) = A(a, b) - G(a, b)$$

and

$$M_{SN_1}(a, b) = S(a, b) - N_1(a, b).$$

The convexity of the measures $M_{SH}(a, b)$, $M_{AG}(a, b)$ and $M_{SN_1}(a, b)$ can be seen in Taneja [4]

Remark 4.2. *The following inequalities hold:*

$$M_{SH}(a, b) \leq \frac{9}{5}M_{SL}(a, b) \leq 3M_{AG}(a, b) \leq \frac{9}{2}M_{AL}(a, b) \quad (4.4)$$

and

$$M_{SA}(a, b) \leq \frac{3}{4}M_{SN_3}(a, b) \leq \frac{2}{3}M_{SN_1}(a, b) \leq \frac{3}{5}M_{SL}(a, b) \quad (4.5)$$

Proof. We know that the following inequalities hold:

$$M_{SA}(a, b) \leq \frac{1}{3}M_{SH}(a, b) \leq \frac{1}{2}M_{AH}(a, b) \leq \frac{1}{2}M_{SG}(a, b) \leq M_{AG}(a, b). \quad (4.6)$$

and

$$M_{SA}(a, b) \leq \frac{3}{4}M_{SN_3}(a, b) \leq \frac{2}{3}M_{SN_1}(a, b). \quad (4.7)$$

The proof of the inequalities (4.6) and (4.7) can be seen in Taneja [4].

Inequalities (4.6) together with (4.1) and (4.2) give (4.4). Inequalities (4.7) together with (4.3) give (4.5). Still we need to show that

$$\frac{9}{5}M_{SL}(a, b) \leq 3M_{AG}(a, b), \quad (4.8)$$

i.e.,

$$M_{SL}(a, b) \leq \frac{5}{3}M_{AG}(a, b)$$

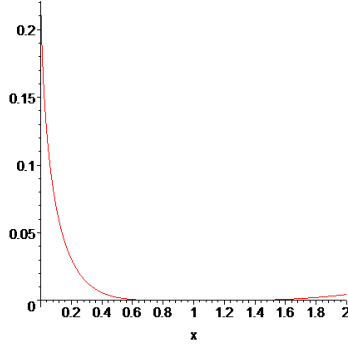
In order to show this let us consider the difference

$$\begin{aligned} T_1(a, b) &= \frac{5}{3}M_{AG}(a, b) - M_{SL}(a, b) \\ &= \frac{5A(a, b) - 5G(a, b)}{3} - S(a, b) + L(a, b) = a f_{T_1}\left(\frac{b}{a}\right), \end{aligned}$$

where

$$f_{T_1}(x) = \frac{5(x+1)}{6} - \frac{5\sqrt{x}}{3} - \frac{\sqrt{2x^2+2}}{2} + \frac{x-1}{\ln x}, \quad x \neq 1, \quad x > 0$$

The graph of the function $f_{T_1}(x)$ is given by



From the above graph it is clear that $T_1(a, b) \geq 0$. Proving the inequalities (4.8), consequently, proving (4.4). \square

Remark 4.3. *The following inequalities hold*

$$\begin{aligned} L(a, b) &\leq \frac{S(a, b) + 9L(a, b)}{10} \leq \frac{2N_3(a, b) + 3L(a, b)}{5} \\ &\leq \frac{5A(a, b) + 7L(a, b)}{12} \leq N_1(a, b) \leq \frac{5N_2(a, b) + L(a, b)}{6}. \end{aligned} \quad (4.9)$$

Proof. Simplifying the inequalities (3.4) and (3.5) given in Theorem 3.1, we get

$$\frac{S(a, b) + 5L(a, b)}{6} \leq \frac{5A(a, b) + 7L(a, b)}{12} \leq N_1(a, b) \leq \frac{5N_2(a, b) + L(a, b)}{6} \quad (4.10)$$

and

$$\frac{S(a, b) + 9L(a, b)}{10} \leq \frac{2N_3(a, b) + 3L(a, b)}{5} \leq N_1(a, b) \quad (4.11)$$

respectively.

Combining the above two inequalities (4.10) and (4.11), we can get (4.9) provided the following hold

$$\frac{2N_3(a, b) + 3L(a, b)}{5} \leq \frac{5A(a, b) + 7L(a, b)}{12} \quad (4.12)$$

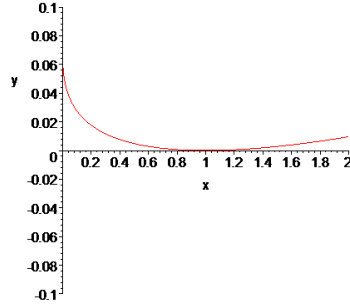
In order to prove inequalities (4.12), let us consider the difference

$$\begin{aligned} T_2(a, b) &= \frac{5A(a, b) + 7L(a, b)}{12} - \frac{2N_3(a, b) + 3L(a, b)}{5} \\ &= \frac{25A(a, b) - 24N_3(a, b) - L(a, b)}{60} = a f_{T_2} \left(\frac{b}{a} \right), \end{aligned}$$

where

$$f_{T_2}(x) = \frac{5}{12} \left(\frac{x+1}{2} \right) - \frac{1}{60} \left(\frac{x-1}{\ln x} \right) - \frac{2}{5} \left(\frac{(1+\sqrt{x})(\sqrt{2x+2})}{4} \right), \quad x \neq 1, \quad x > 0$$

The graph of the function $f_{T_2}(x)$, $x > 0$, $x \neq 1$ is given by



From the above graph we observe that the function $f_{T_2}(x)$, $x > 0$, $x \neq 1$, is nonnegative with $f_{T_2}(1) = 0$. Thus we conclude that the validity of the expression (4.4) proving the inequalities (4.1). \square

Remark 4.4. *There is no relation between the expressions*

$$\frac{S(a, b) + 5L(a, b)}{6}$$

and

$$\frac{2N_3(a, b) + 3L(a, b)}{5}.$$

Proof. This we shall show by simple example. Let us write the difference

$$T_3(a, b) = \frac{S(a, b) + 5L(a, b)}{6} - \frac{2N_3(a, b) + 3L(a, b)}{5} = a f_{T_3} \left(\frac{b}{a} \right)$$

where

$$f_{T_3}(x) = \frac{\sqrt{2+2x^2}}{12} + \frac{7}{30} \left(\frac{x-1}{\ln x} \right) - \frac{(1+\sqrt{x})\sqrt{2x+2}}{10}, \quad x \neq 1, \quad x > 0$$

By simple calculations, we have

$$f_{T_3}(0.00001) = -0.00337512758 \text{ and } f_{T_3}(1.1) = 0.0001321351.$$

Thus we have two values, one negative and another positive proving that we can't have relation between these two measures. \square

Remark 4.5. *The followings hold*

$$N_2(a, b) \leq \frac{5N_2(a, b) + L(a, b)}{6} \leq N_3(a, b). \quad (4.13)$$

Proof. In view of (4.10), it is sufficient to show that

$$N_3(a, b) - \frac{5N_2(a, b) + L(a, b)}{6} \geq 0$$

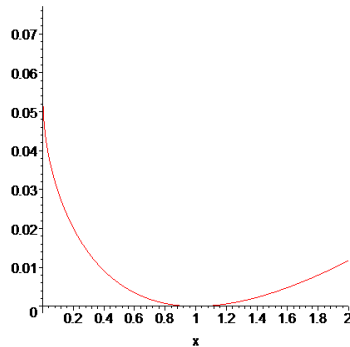
Let us write

$$T_4(a, b) = N_3(a, b) - \frac{5N_2(a, b) + L(a, b)}{6} = a f_{T_3} \left(\frac{b}{a} \right),$$

where

$$f_{T_4}(x) = \frac{(1+\sqrt{x})(\sqrt{2x+2})}{4} - \frac{5(1+\sqrt{x}+x)}{18} - \frac{1}{6} \frac{x-1}{\ln x}, \quad x \neq 1, \quad x > 0$$

The graph of the function $f_{T_4}(x)$ is given by



From the graph above it is clear that $T_4(a, b) \geq 0$. Proving the inequalities (4.13). \square

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